Journal of Engineering Mathematics, Vol. 7, No. 4, October 1973
Noordhoff International Publishing-Leyden
Printed in The Netherlands

Finite waiting space bulk queueing systems

T. P. BAGCHI* AND J. G. C. TEMPLETON

Department of Industrial Engineering, University of Toronto, Canada

(Received August 7, 1972)

SUMMARY
This paper extends earlier work on the stationary queue length distribution of a bulk service system with finite waiting space by considering two queueing systems. The first system incorporates the feature of batch arrivals with group service; it has compound Poisson input, general service times and a single server with variable batch capacity. The second system has individually arriving customers with Erlangian interarrival time distribution, general service times and a single server with variable batch capacity.
The financial support of the National Research Council of Canada, under Research Grants No. 3-644-189-60; NRC-A5639 and No. 3-641-189-10; NRC-A2796, is acknowledged.

1. Introduction

Singh [2] has studied the stationary behaviour of a finite waiting space queueing system with simple Poisson input, general service times, and a single server with variable batch capacity. In the present paper we employ a slightly altered "imbedding" from his, but essentially the same procedure, to consider two different queueing systems. The first system has compound Poisson input, general service times and a single server with variable batch capacity. This is a bulk queueing system, incorporating the feature of batch arrivals with group service, and is denoted by $M^{(X)}/G^{(Y)}/1/(N)$. The second system has individually arriving customers with Erlangian interarrival time distribution, general service times and a single server with variable batch capacity. Unless otherwise specified, the notations employed in this paper will be identical with those of Singh.

2. $M^{(X)}/G^{(Y)}/1/(N)$ Queueing system

This bulk queueing system can be characterised as follows:

(i) Customers arrive in batches that are of variable size. The batches arrive one at a time in a Poisson process with parameter λ. The probability that n batches arrive in a time interval $(0, t)$ equals

$$\frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

(ii) If $p_j^{(n)}$ denotes the probability that n arriving batches bring a total of j customers, then the probability that j customers arrive in a time interval $(0, t)$, assuming that each arriving batch brings at least one customer, may be written as

$$\sum_{n=1}^{j} \frac{e^{-\lambda t}(\lambda t)^n}{n!} p_j^{(n)}.$$

If, for instance, the sizes of the arriving batches are distributed geometrically, with $\Pr\{\text{batch size} = v\} = (1-p)p^{v-1}$, $v = 1, 2, 3, \ldots$ then

$$p_j^{(n)} = \binom{j-1}{n-1} p^{j-n}(1-p)^n, \qquad j \geqq n > 0.$$

* Now at Imperial Oil Ltd., Sarnia, Ontario, Canada.

(iii) The customers are served in batches of variable capacity, the maximum service capacity for the server being s. The server stays continuously busy and begins a service period immediately after the completion of the preceding period. Also, he is "intermittently available", in that arrivals joining the system after the commencement of a given service period must wait for the commencement of the next service period. (Bailey's [1] model incorporates a similar rule.) Let $t_1, t_2, \ldots$ be the instants of completion of a sequence of service periods and let v_n denote the service time $(t_n - t_{n-1})$. We assume that $\{v_n\}$ is a sequence of independent and identically distributed random variables with a common distribution function $G(t)$ $(0 \leqq t < \infty)$. Also, we assume that v_n, $n=1, 2, 3, \ldots$ are independent of the arrival process.

(iv) Let $s - Y_n$ be the capacity for service ending at t_{n+1} $(n=0, 1, 2, \ldots)$. We assume that $\{Y_n\}$ are independent and identically distributed random variables, also independent of the arrival process. Let

$$\begin{aligned} \Pr\{Y_n = r\} &= b_r\,, \qquad 0 \leqq r \leqq s \\ &= 0 \qquad\quad\; r > s\,. \end{aligned}$$

For the service period starting immediately after t_n, the server takes a number of customers equal to $\min(s - Y_n$, whole queue length). Let

$$B_j = \sum_{r=0}^{j} b_r$$

and

$$B_s(x) = \sum_{r=0}^{s} b_r x^r$$

with $B_0 = b_0$.

(v) The waiting room has a fixed capacity of N customers, including those in service. An arrival finding the system full balks by assumption, and an arrival after joining the system does not renege.

Singh's model is an $M/G^{(Y)}/1/(N+1)$ queue in which state E_{N+1} is never occupied at a time (t_n+0) just after a departure. The present model is an $M^{(X)}/G^{(Y)}/1/(N)$ queue in which state E_N may be occupied at a time (t_n+0) just after a departure, because the service capacity $(s - Y_n)$ of the system may be zero. Therefore both models give rise to imbedded Markov chains with the same state space $\{E_0, E_1, \ldots, E_N\}$.

3. Analysis of the $M^{(X)}/G^{(Y)}/1/(N)$ system

The analysis proceeds with the specification of a discrete parameter Markov chain which is imbedded in the continuous time parameter queueing process. We define the state of the system as E_j when there are j customers in the system immediately following a service completion. The transition probabilities of the imbedded Markov chain are defined as follows:

$$\begin{aligned} r_{ij} &= \Pr\{j \text{ customers in system at } (t_{n+1}+0) \,|\, i \text{ customers in system at } (t_n+0)\} \\ &= \Pr\{\text{next state is } E_j \,|\, \text{previous state was } E_i\} \end{aligned}$$

The equilibrium (stationary) probabilities of the chain are defined as

$$p_j = \Pr\{\text{the system is in state } E_j\}\,, \qquad j = 0, 1, 2, \ldots, N\,,$$

and an associated probability generating function is written as

$$P(x) = \sum_{j=0}^{N-1} p_j x^j\,.$$

Since customers may arrive in batches, the probability that a given number of customers arrive during a service period in the present case is different from the corresponding probability specified by Singh. Accordingly, if X_n denotes the number of customers arriving during the service period ending at t_n, then the distribution of X_n will be given by

$$\Pr\{X_n = j\} = k_j = \int_0^\infty \sum_{i=1}^{j} \frac{e^{-\lambda t}(\lambda t)^i}{i!} p_j^{(i)} dG(t),$$

where $p_j^{(i)}$ denotes the probability that i arriving batches bring a total of j customers, as stated earlier. An associated generating function is defined as

$$K(x) = \sum_{j=0}^{N-1} k_j x^j .$$

When the above notations are employed, the transition probability matrix for the $M^{(X)}/G^{(Y)}/1/(N)$ queueing system becomes identical with Table 1 of Singh, after changing that table so that $r_{2N} = l_N B_{s-2} + l_{N-1} b_{s-1} + l_{N-2} b_s$. Likewise, the equations determining the equilibrium probabilities become

$$\begin{aligned} p_j = {} & k_j p_0 + (k_j B_{s-1} + k_{j-1} b_s) p_1 + (k_j B_{s-2} + k_{j-1} b_{s-1} + k_{j-2} b_s) p_2 + \ldots \\ & + (k_j B_1 + k_{j-1} b_2 + \ldots + k_{j-s+1} b_s) p_{s-1} \\ & + (k_j b_0 + k_{j-1} b_1 + \ldots + k_{j-s} b_s) p_s + \ldots \\ & + (k_{j-N+s+1} b_0 + k_{j-N+s} b_1 + \ldots + k_{j-N+1} b_s) p_{N-1} \\ & + (k_{j-N+s} b_0 + k_{j-N+s-1} b_1 + \ldots + k_{j-N} b_s) p_N \\ & \text{for } j = 0, 1, 2, \ldots, N-1 \end{aligned}$$

and

$$\begin{aligned} p_N = {} & l_N p_0 + (l_N B_{s-1} + l_{N-1} b_s) p_1 + (l_N B_{s-2} + l_{N-1} b_{s-1} + l_{N-2} b_s) p_2 + \ldots \\ & + (l_{s+1} b_0 + l_s b_1 + \ldots + l_1 b_s) p_{N-1} + (l_s b_0 + \ldots + l_0 b_s) p_N , \end{aligned}$$

where

$$l_r = k_r + k_{r+1} + k_{r+2} + \ldots$$

Following Singh (pages 244–6) one obtains

$$Q(x) = \frac{\sum_{i=0}^{s-1} p_i \{x^s B_{s-i} - x^i B_{s-i}(x)\}}{x^s/K(x) - B_s(x)}$$

From this the probabilities $p_0, p_1, \ldots, p_N$, which specify the stationary behaviour of the $M^{(X)}/G^{(Y)}/1/(N)$ queueing system may be evaluated, when a count of the number of customers present in the system is made immediately following every service completion. Stationary results due to Singh for the $M/G^{(Y)}/1/(N+1)$ queue are obtained from $Q(x)$ above if the function $K(x)$ is defined as in equation (3) of Singh.

4. $E_l/G^{(Y)}/1/(N)$ queueing system

This queueing system can be described as follows:

(i) Customers arrive one by one, the interarrival times being independent and identically distributed random variables with an Erlangian distribution of order l and mean l/λ. Each of the arriving customers may be assumed to pass through l different stages, the durations of the stages being mutually independent random variables with the distribution $\lambda e^{-\lambda t} dt (0 \leq t < \infty)$.

(ii) The customers are served in batches of variable capacity, the maximum service capacity for the server being s. The server is continuously busy and is "intermittently available" as assumed in the $M^{(X)}/G^{(Y)}/1/(N)$ case above.

(iii) Let $s - Y_n$ be the capacity for service ending at t_{n+1} $(n = 0, 1, \ldots)$. It is assumed that $\{Y_n\}$ are independent and identically distributed random variables, also independent of the arrival process. Let

$$\Pr\{Y_n = r\} = b_r, \qquad 0 \leqq r \leqq s$$
$$= 0 \qquad r > s.$$

For the service starting after t_n the server takes $\min(s - Y_n$, whole queue length). Let

$$B_j = \sum_{r=0}^{j} b_r$$

and

$$B_s(x) = \sum_{r=0}^{s} b_r x^r$$

with $B_0 = b_0$.

(iv) The waiting room has a fixed capacity of N customers, including those in service. An arrival finding the system full balks by assumption, and an arrival after joining the system does not renege.

The queueing process $\{Q(t)\}$ in this system is non-Markovian in general. It is possible, however, to obtain a Markov chain that is imbedded in the process $\{R(t)\}$, which is the number of stages completed by the customers who are in the system at time t. We note that when a customer departs from the queueing system, the current value of $R(t)$ is reduced by l. We define the state of the system as E_j when there are j stages in the system immediately following a service completion. The transition probabilities of the imbedded Markov chain are defined as follows:

$$r_{ij} = \Pr\{\text{next state is } E_j | \text{previous state was } E_i\}.$$

The equilibrium probabilities of the system are defined as

$$p_j = \Pr\{\text{the system is in state } j\}, \quad j = 0, 1, 2, \ldots, Nl + l - 1.$$

Let X_n be the number of stages arriving during a service period ending at t_n. Then the distribution of X_n is given by

$$\Pr\{X_n = j\} = k_j = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!} dG(t)$$

We next introduce a new set of probabilities $\{a_i\}$ that are associated with the number of stages leaving the queueing system as customers are served in batches of variable size. Let

$$a_i = b_r, \quad i = lr, \qquad 0 \leqq r \leqq s$$
$$= 0 \quad \text{otherwise}.$$

Let

$$A_j = \sum_{i=0}^{j} a_i$$

with $A_0 = b_0$. The transition probability matrix for the system may now be constructed. It is a $(N+1)l \times (N+1)l$ stochastic matrix, structurally similar to Table 1 of Singh. Since the state space $\{E_j\}$ is finite, the system must possess a unique equilibrium distribution. The equations determining the equilibrium probabilities become

$$\begin{aligned} p_j = {} & k_j p_0 + k_{j-1} a_{ls} p_1 + k_{j-2} a_{ls} p_2 + \ldots + k_{j-s+1} a_{ls} p_{s-1} + k_{j-s} a_{ls} p_s + \ldots \\ & + p_l (k_j A_{l(s-1)} + k_{j-l} a_{ls}) + p_{l+1} k_{j-l-1} a_{ls} + \ldots \\ & + p_{2l} (k_j A_{l(s-2)} + k_{j-l} a_{l(s-1)} + k_{j-2l} a_{ls}) \\ & + \ldots \\ & + p_{sl} (k_j A_{l(s-s)} + k_{j-l} a_l + k_{j-2l} a_{2l} + \ldots + k_{j-sl} a_{sl}) \\ & + \ldots \\ & + p_{Nl} (k_{j-l(N-s)} A_{l(s-s)} + k_{j-l(N-s+1)} a_l + k_{j-l(N-s+2)} a_{2l} + \ldots + k_{j-Nl} a_{sl}) \end{aligned}$$

$$
\begin{aligned}
&+p_{Nl+1}(k_{j-l(N-s)-1}A_{l(s-s)}+k_{j-l(N-s+1)-1}a_l+k_{j-l(N-s+2)-1}a_{2l}+\ldots+k_{j-Nl-1}a_{sl})\\
&+\ldots\\
&+p_{Nl+l-1}(k_{j-l(N-s+1)+1}A_{l(s-s)}+k_{j-l(N-s+2)+1}a_l+\\
&\qquad\qquad+k_{j-l(N-s+3)+1}a_{2l}+\ldots+k_{j-l(N+1)+1}a_{sl}),\\
&\text{for}\quad j=0,1,2,\ldots,(N+1)l-2.
\end{aligned}
$$

The above provides $Nl+l-1$ linear simultaneous equations, involving $(N+1)l$ unknowns. The solution of this system of equations, combined with $\sum_{j=0}^{Nl+l-1} p_j=1$ yields the equilibrium probabilities $\{p_j\}$. A numerical approach to this solution is in practice satisfactory; it avoids the determination of the zeros of certain polynomials and of the coefficients in a power series that determine the desired state probabilities. If R_n now denotes the number of completed stages at an arbitrary departure epoch (in this steady state) marked by n, then the corresponding number of customers in the system will be given by Q_n, when Q_n is the largest integer such that $lQ_n \leqq R_n$.

Note added in proof:
Lwin and Ghosal [4] have studied the $M/G^b/1/(N+1)b$ queueing model, with fixed service capacity b, using the imbedded Markov chain method and obtaining a generating function which can be found from Singh's results. Gaur [3] obtains time-dependent and stationary solutions for the queue length distribution in a $M^X/M^Y/1/(N)$ queue, subject to constraints on the size of arrival batches and service batches.

REFERENCES

[1] N. T. J. Bailey, On Queueing Processes with Bulk Service, *Journal of the Royal Statistical Society, Series B*, 16, pp. 80–87, 1954.

[2] V. P. Singh, Finite Waiting Space Bulk Service System, *Journal of Engineering Math.*, 5, No. 4, pp. 241–248, 1971; Addendum, *Journal of Engineering Math.*, 6, No. 1, pp. 85–88, 1972.

[3] R. S. Gaur, A Limited Queueing Problem with Any Number of Arrivals and Departures, *Revue Française d'Automatique, Informatique et Recherche Opérationelle*, 6, 87–93, V-2, October 1972.

[4] T. Lwin and A. Ghosal, On the Finite Capacity Treatment of Bailey's Queueing Model, *Calcutta Statist. Assoc. Bull.*, 20, 67–76, 1971.